DAY THIRTY NINE

Mock Test 2

(Based on Complete Syllabus)

Instructions

1. This question paper contains of 30 Questions of Mathematic, divided into two Sections :
   Section A Objective Type Questions and **Section B** Numerical Type Questions.
2. Section A contains 20 Objective questions and all Questions are compulsory (**Marking Scheme** : Correct +4, Incorrect −1).
3. Section B contains 10 Numerical value questions out of which only 5 questions are to be attempted (**Marking Scheme** : Correct +4, Incorrect 0).

Section A : Objective Type Questions

1 The average weight of students in a class of 35 students is 40 kg. If the weight of the teacher be included, then average rises by $\frac{1}{2}$ kg; the weight of the teacher is

(a) 40.5 kg (b) 50 kg (c) 41 kg (d) 58 kg

2 Let $f(x) = \frac{1}{[\sin x]}$, $[\cdot]$ being the greatest integer function, then

(a) $f(x)$ is not continuous, where $x \in (2n\pi, 2n\pi + \pi), n \in I$
(b) $f(x)$ is differentiable at $x = \frac{\pi}{4}$
(c) $f(x)$ is differentiable at $x = \frac{\pi}{2}$
(d) None of the above

3 If $\int f(x) \sin x \cos x \, dx = \frac{1}{2(b^2 - a^2)} \log [f(x)] + C$, then $f(x)$ is equal to

(a) $\frac{1}{a^2 \sin^2 x + b^2 \cos^2 x}$ (b) $\frac{1}{a^2 \sin^2 x - b^2 \cos^2 x}$
(c) $\frac{1}{a^2 \cos^2 x - b^2 \sin^2 x}$ (d) $\frac{1}{a^2 \cos^2 x + b^2 \sin^2 x}$

4 The solution of the differential equation $\frac{dy}{dx} = \frac{y}{x} + \frac{\phi\left(\frac{y}{x}\right)}{\phi'\left(\frac{y}{x}\right)}$ is

(a) $x\phi\left(\frac{y}{x}\right) = k$ (b) $\phi\left(\frac{y}{x}\right) = kx$
(c) $y\phi\left(\frac{y}{x}\right) = k$ (d) $\phi\left(\frac{y}{x}\right) = ky$

5 If $a \le \tan^{-1} x + \cot^{-1} x + \sin^{-1} x \le b$, then $a + b$ is equal to

(a) 0 (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{4}$ (d) π

6 The two consecutive terms in the expansion of $(3 + 2x)^{74}$ whose coefficients are equal, are

(a) 11, 12 (b) 7, 8
(c) 30, 31 (d) None of these

7 A parabola is drawn with its focus at (3, 4) and vertex at the focus of the parabola $y^2 - 12x - 4y + 4 = 0$. The equation of the parabola is

(a) $y^2 - 8x - 6y + 25 = 0$ (b) $y^2 - 6x + 8y - 25 = 0$
(c) $x^2 - 6x - 8y + 25 = 0$ (d) $x^2 + 6x - 8y - 25 = 0$

8 If p, p' denote the lengths of the perpendiculars from the focus and the centre of an ellipse with semi-major axis of length a respectively on a tangent to the ellipse and r denotes the focal distance of the point, then

(a) $ap' = rp + 1$ (b) $rp = ap'$

(c) $ap = rp' + \frac{1}{\sqrt{3}}$ (d) $ap = rp'$

9 The equation of the locus of the pole with respect to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, of any tangent line to the auxiliary circle, is the curve $\frac{x^2}{a^4} + \frac{y^2}{b^4} = \lambda^2$, where

(a) $\lambda^2 = a^2$ (b) $\lambda^2 = \frac{1}{a^2}$

(c) $\lambda^2 = b^2$ (d) $\lambda^2 = \frac{1}{b^2}$

10 The solution set for $\frac{(2x+3)(4-3x)^3(x-4)}{(x-2)^2 x^5} \le 0$

(a) $\left(-\infty, -\frac{3}{2}\right) \cup \left(0, \frac{4}{3}\right) \cup (4, \infty)$

(b) $\left(-\frac{3}{2}, 0\right) \cup \left(\frac{4}{3}, 4\right)$

(c) $(-\infty, 0) \cup (2, \infty)$

(d) None of the above

11 $f(x) = x - \cot^{-1} x - \log(x + \sqrt{1+x^2})$ is increasing in

(a) $(-\infty, \infty)$ (b) $(-\infty, 2)$

(c) $(2, 5)$ (d) $(-\infty, 10)$

12 The contrapositive of the statement, "if x is a prime number and x divides ab, then x divides a or x divides b", can be symbolically represented using logical connectives on appropriately defined statements p, q, r and s as

(a) $(\sim r \vee \sim s) \to (\sim p \wedge \sim q)$

(b) $(r \wedge s) \to (\sim p \wedge \sim q)$

(c) $(\sim r \wedge \sim s) \to (\sim p \vee \sim q)$

(d) $(r \vee s) \to (\sim p \vee \sim q)$

13 The area bounded by the lines $y = 2$, $x = 1$, $x = a$ and the curve $y = f(x)$, which cuts the last two lines above the first line for all $a > 1$, is equal to $\frac{2}{3}[(2a)^{3/2} - 3a + 3 - 2\sqrt{2}]$..

Then, $f(x)$ equals

(a) $2\sqrt{2x}, x > 1$ (b) $\sqrt{2x}, x > 1$

(c) $2\sqrt{x}, x > 1$ (d) None of these

14 Area of a triangle with vertices $(a, b), (x_1, y_1)$ and (x_2, y_2), where a, x_1 and x_2 are in GP with common ratio r and b, y_1 and y_2 are in GP with common ratio s, is given by

(a) $\frac{1}{2} ab(r-1)(s-1)(s-r)$

(b) $\frac{1}{2} ab(r+1)(s+1)(s-r)$

(c) $ab(r-1)(s-1)(s-r)$

(d) None of the above

15 The equation of perpendicular bisectors of sides AB and AC of a ΔABC are $x - y + 5 = 0$ and $x + 2y = 0$, respectively. If the coordinates of vertex A are $(1, -2)$, then equation of BC is

(a) $14x + 23y - 40 = 0$ (b) $14x - 23y + 40 = 0$

(c) $23x + 14y - 40 = 0$ (d) $23x - 23y + 40 = 0$

16 If $|z - 25i| \le 15$, then $|\max \text{amp}(z) - \min \text{amp}(z)|$ is equal to

(a) $\cos^{-1}\left(\frac{3}{5}\right)$ (b) $\pi - 2\cos^{-1}\left(\frac{3}{5}\right)$

(c) $\frac{\pi}{2} + \cos^{-1}\left(\frac{3}{5}\right)$ (d) $\sin^{-1}\left(\frac{3}{5}\right) - \cos^{-1}\left(\frac{3}{5}\right)$

17 If $ABCD$ is a cyclic quadrilateral such that $12\tan A - 5 = 0$ and $5\cos B + 3 = 0$, then the quadratic equation whose roots are $\cos C$ and $\tan D$ is

(a) $39x^2 + 88x + 48 = 0$

(b) $39x^2 - 16x - 48 = 0$

(c) $39x^2 - 88x + 48 = 0$

(d) None of the above

18 A mirror and a source of light are situated at the origin O and a point on OX, respectively. A ray of light from the source strikes the mirror and is reflected. If the DR's of the normal to the plane of mirror are 1, –1, 1, then DC's for the reflected ray are

(a) $\frac{1}{3}, \frac{2}{3}, \frac{2}{3}$ (b) $-\frac{1}{3}, \frac{2}{3}, \frac{2}{3}$

(c) $-\frac{1}{3}, -\frac{2}{3}, -\frac{2}{3}$ (d) $-\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}$

19 If both roots of the equation $4x^2 - 2x + a = 0$, $a \in R$, lies in the interval $(-1, 1)$, then

(a) $-2 < a \le \frac{1}{4}$ (b) $-6 \le a \le \frac{1}{4}$

(c) $a > \frac{1}{4}$ (d) $a < -2$

20 Let λ and θ be real numbers. Then, the set of all values of λ for which the system of linear equations

$$\lambda x + (\sin\theta)y + (\cos\theta)z = 0$$
$$x + (\cos\theta)y + (\sin\theta)z = 0$$
$$-x + (\sin\theta)y - (\cos\theta)z = 0$$

has a non-trivial solution, is

(a) $[0, \sqrt{2}]$ (b) $[-\sqrt{2}, 0]$

(c) $[-\sqrt{2}, \sqrt{2}]$ (d) None of these

Section B : Numerical Type Questions

21 If the value of integral $\left|\int_0^\pi [2\sin x]\,dx\right|$, where $[x]$ denotes greatest integer function, is $\frac{2\pi}{T}$, then value of T is equal to

22 Suppose $A_1, A_2, \ldots, A_{30}$ are thirty sets, each having 5 elements and $B_1, B_2, \ldots, B_n$ are n sets each with 3 elements, let $\bigcup_{i=1}^{30} A_i = \bigcup_{j=1}^{n} B_j = S$ and each element of

S belongs to exactly 10 of the A_i's and exactly 9 of the B_j's Then, n is equal to

23 An experiment succeeds twice as often as it fails. Then, the probability that in the next 4 trials there will be atleast 2 successes, is p/q, then value of pq is equal to

24 A variable plane at a constant distance P, from origin cuts axes at A, B and C, the locus of centroid of tetrahedron $OABC$ is $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{k^2}{p^2}$, then value of k^2 is equal to

25 A parallelogram is constructed on the vectors $\mathbf{a} = 3\alpha - \beta$, $\mathrm{b} = \alpha + 3\beta$, if $|\alpha| = |\beta| = 2$ and angle between α and β is $\frac{\pi}{3}$, length of a diagonal of the parallelogram is $p\sqrt{3}$, then value of p is equal to ....

26 Let $A = \begin{bmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{bmatrix}$, where $0 \le \theta \le 2\pi$.

Then, largest value of $|A|$ is

27 The value of the expression $\frac{\sin^3 x}{1+\cos x}+\frac{\cos^3 x}{1-\sin x}$ is $\frac{1}{\sqrt{a}}\cos\left(\frac{\pi}{4}-x\right)$, then value of a is equal to ....

28 Number of ways in which 5 boys and 4 girls can be arranged on a circular table such that no two girls sit together and two particular boys are always together is ....

29 If $S_n = 1+\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^{n-1}}$ and $2-S_n < \frac{1}{100}$, then the least value of n must be ....

30 If X and Y are independent binomial variates $B\left(5, \frac{1}{2}\right)$ and $B\left(7, \frac{1}{2}\right)$. If the value of $P(X+Y=3)$ is λ, then the value of 4096λ must be ....

Hints and Explanations

1 *(d)* Let the weight of teacher be x kg, then

$$40 + \frac{1}{2} = \frac{35 \times 40 + x}{35 + 1}$$

$$\Rightarrow \frac{81}{2} \times 36 = 35 \times 40 + x$$

$$\Rightarrow 81 \times 18 = 1400 + x$$

$$\Rightarrow 1458 = 1400 + x \Rightarrow x = 58$$

Hence, the weight of teacher is 58 kg.

2 *(a)* We have, $f(x) = \frac{1}{[\sin x]}$

Clearly, $\sin x \notin [0, 1)$

$[\because \text{if } 0 \le \sin x < 1, [\sin x] = 0]$

$$\Rightarrow x \notin [2n\pi, (2n+1)\pi] - (4n+1)\frac{\pi}{2}, n \in I$$

Thus, $f(x)$ is not continuous if $x \in (2n\pi, 2n\pi + \pi), n \in I$.

3 *(a)* Given that, $\int f(x) \sin x \cos x \, dx$

$$= \frac{1}{2(b^2 - a^2)} \log [f(x)] + C$$

On differentiating both sides, we get

$$f(x) \sin x \cos x = \frac{d}{dx}\left\{\frac{\log[f(x)]}{2(b^2 - a^2)} + C\right\}$$

$$\Rightarrow f(x) \sin x \cos x = \frac{1}{2(b^2 - a^2)} \cdot \frac{1}{f(x)} f'(x)$$

$$\Rightarrow 2(b^2 - a^2) \sin x \cos x = \frac{f'(x)}{[f(x)]^2}$$

On integrating both sides, we get

$$-b^2 \cos^2 x - a^2 \sin^2 x = -\frac{1}{f(x)}$$

$$\therefore f(x) = \frac{1}{a^2 \sin^2 x + b^2 \cos^2 x}$$

4 *(b)* Given, $\frac{dy}{dx} - \frac{y}{x} = \frac{\phi\left(\frac{y}{x}\right)}{\phi'\left(\frac{y}{x}\right)}$

$$\Rightarrow \frac{\phi'\left(\frac{y}{x}\right)\left(\frac{x\,dy - y\,dx}{x^2}\right)}{\phi\left(\frac{y}{x}\right)} = \frac{1}{x} dx$$

$$\Rightarrow \int \frac{\phi'\left(\frac{y}{x}\right) d\left(\frac{y}{x}\right)}{\phi\left(\frac{y}{x}\right)} = \int \frac{1}{x} dx + \log k$$

$$\Rightarrow \log \phi\left(\frac{y}{x}\right) = \log x + \log k$$

$$\Rightarrow \phi\left(\frac{y}{x}\right) = kx$$

$\therefore$ Sum of two

5 *(d)* We know, $-\frac{\pi}{2} \le \sin^{-1} x \le \frac{\pi}{2}$

$$\therefore 0 \le \frac{\pi}{2} + \sin^{-1} x \le \pi$$

$$\Rightarrow 0 \le \tan^{-1} x + \cot^{-1} x + \sin^{-1} x \le \pi$$

$\Rightarrow a = 0$ and $b = \pi$

Hence, $a + b = \pi$

6 *(c)* General term of $(3 + 2x)^{74}$ is

$$T_{r+1} = {}^{74}C_r (3)^{74-r} 2^r x^r$$

Let two consecutive terms be T_{r+1} th and T_{r+2} th terms.

According to the question

Coefficient of T_{r+1} = Coefficient of T_{r+2}

$$\Rightarrow {}^{74}C_r 3^{74-r} 2^r = {}^{74}C_{r+1} 3^{74-(r+1)} 2^{r+1}$$

$$\Rightarrow \frac{{}^{74}C_{r+1}}{{}^{74}C_r} = \frac{3}{2}$$

$$\Rightarrow \frac{74 - r}{r + 1} = \frac{3}{2}$$

$$\Rightarrow 148 - 2r = 3r + 3$$

$$\therefore r = 29$$

Hence, two consecutive terms are 30 and 31.

7 *(c)* Given, equation can be rewritten as $(y - 2)^2 = 12x$

Here, vertex and focus are (0, 2) and (3, 2).

$\therefore$ Vertex of the required parabola is (3, 2) and focus is $(3, 4)$.

The axis of symmetry is $x = 3$ and latusrectum $= 4 \cdot 2 = 8$

Hence, required equation is

$$(x - 3)^2 = 8(y - 2)$$

$$\Rightarrow x^2 - 6x - 8y + 25 = 0$$

8 *(d)* Tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at $P(a \cos \theta, b \sin \theta)$ is

$$\frac{x}{a} \cos \theta + \frac{y}{b} \sin \theta = 1 \quad \text{...(i)}$$

Now, p = perpendicular distance from focus $S(ae, 0)$ to the line (i)

$$= \frac{\left|\frac{ae}{a} \cos \theta + 0 - 1\right|}{\sqrt{\frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2}}}$$

$$= \frac{1 - e \cos \theta}{\sqrt{\frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2}}} \quad \text{...(ii)}$$

Also, p' = perpendicular distance from centre (0, 0) to the line (i) .

$$= \frac{1}{\sqrt{\frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2}}} \quad \text{...(iii)}$$

Again, $r = SP = a(1 - e \cos \theta)$

$$\therefore ap = \frac{a - ae \cos \theta}{\sqrt{\frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2}}} = rp'$$

[using Eqs. (ii) and (iii)]

9 *(b)* The equation of the auxiliary circle is $x^2 + y^2 = a^2$.

Let (h, k) be the pole, then equation of the polar of (h, k) with respect to the given ellipse is

$$\frac{hx}{a^2} + \frac{ky}{b^2} = 1$$

Since, this is tangent to the circle.

$$\Rightarrow \frac{|0 + 0 - 1|}{\sqrt{\left(\frac{h}{a^2}\right)^2 + \left(\frac{k}{b^2}\right)^2}} = \pm a$$

$$\Rightarrow \frac{h^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{a^2}$$

Hence, locus of (h, k) is

$$\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{a^2}$$

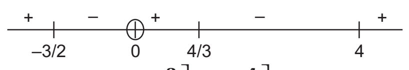
10 *(d)* Given inequality can be rewritten as

$$\frac{(2x + 3)(3x - 4)^3(x - 4)}{(x - 2)^2 x^5} \ge 0$$

$$\Rightarrow (2x + 3)(3x - 4)^3 (x - 4)(x - 2)^2 x^5 \ge 0, x \ne 0, 2$$

$$\Rightarrow (2x + 3)(3x - 4)^3(x - 4)x^5 \ge 0; \; x \ne 0, 2 \; [\because (x - 2)^2 > 0]$$

$$x = -\frac{3}{2}, \frac{4}{3}, 4, 0$$



$$\Rightarrow x \in \left(-\infty, -\frac{3}{2}\right] \cup \left(0, \frac{4}{3}\right] \cup [4, \infty)$$

11 *(a)* Given that,

$$f(x) = x - \cot^{-1} x - \log(x + \sqrt{1 + x^2})$$

On differentiating w.r.t. *x*, we get

$$f'(x) = 1 + \frac{1}{1 + x^2} - \frac{1}{(x + \sqrt{1 + x^2})}\left(1 + \frac{x}{\sqrt{1 + x^2}}\right)$$

$$= 1 + \frac{1}{1 + x^2} - \frac{1}{\sqrt{1 + x^2}}$$

$$= \frac{1 + x^2 + 1 - \sqrt{1 + x^2}}{1 + x^2}$$

$$= \frac{2 + x^2 - \sqrt{1 + x^2}}{1 + x^2}$$

So, $f(x)$ is an increasing function in $(-\infty, \infty)$.

12 *(c)* Let $p = x$ is a prime number

$q = x$ divides ab

$r = x$ divides a

and $s = x$ divides b

The given statement becomes in logical form is $p \wedge q \to r \vee s$

Its contrapositive is

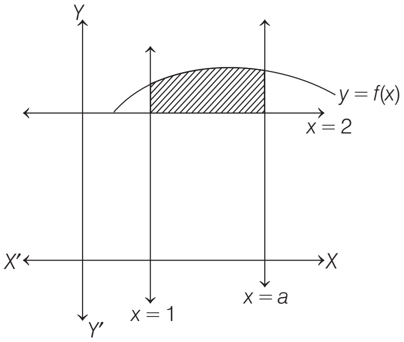
$$\sim(r \vee s) \to \sim(p \wedge q)$$

$$\Rightarrow \quad (\sim r \wedge \sim s) \to (\sim p \vee \sim q)$$

13 *(a)* We have,

$$\int_1^a [f(x) - 2]dx = \frac{2}{3}[(2a)^{3/2} - 3a + 3 - 2\sqrt{2}]$$

On differentiating both sides w.r.t. *a*, we get



$$f(a) - 2 = \frac{2}{3}\left[\frac{3}{2}\sqrt{2a} \cdot 2 - 3\right]$$

$$\Rightarrow \quad f(a) - 2 = 2\sqrt{2a} - 2$$

$$\Rightarrow \quad f(a) = 2\sqrt{2a}, a > 1$$

$$\Rightarrow \quad f(x) = 2\sqrt{2x}, x > 1$$

14 *(a)* Since, a, x_1 and x_2 are in GP with common ratio r.

$$\therefore \quad x_1 = ar, x_2 = ar^2$$

Also, b, y_1 and y_2 are in GP with common ratio s.

$$\therefore \quad y_1 = bs, y_2 = bs^2$$

The area of triangle is given by

$$\Delta = \frac{1}{2}\begin{vmatrix} a & b & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} a & b & 1 \\ ar & bs & 1 \\ ar^2 & bs^2 & 1 \end{vmatrix}$$

$$= \frac{1}{2}ab\begin{vmatrix} 1 & 1 & 1 \\ r & s & 1 \\ r^2 & s^2 & 1 \end{vmatrix}$$

$$= \frac{ab}{2}\begin{vmatrix} 1 & 0 & 0 \\ r & s - r & 1 - r \\ r^2 & s^2 - r^2 & 1 - r^2 \end{vmatrix}$$

[applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$]

$$= \frac{ab}{2}\{(s - r)(1 - r^2) - (1 - r)(s^2 - r^2)\}$$

$$= \frac{ab}{2}(s - r)(1 - r)\{1 + r - (s + r)\}$$

$$= \frac{ab}{2}(s - r)(1 - r)(1 - s)$$

$$= \frac{ab}{2}(s - r)(r - 1)(s - 1)$$

15 *(a)* Let the coordinates of B and C are (x_1, y_1) and (x_2, y_2) respectively.

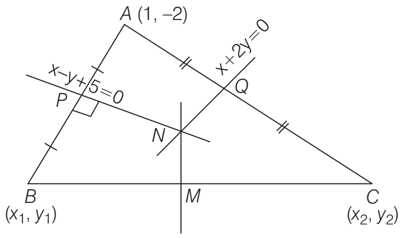
Then, $P\left(\frac{x_1 + 1}{2}, \frac{y_1 - 2}{2}\right)$ lies on perpendicular bisector $x - y + 5 = 0$.

$$\therefore \quad \frac{x_1 + 1}{2} - \frac{y_1 - 2}{2} = -5$$

$$\Rightarrow \quad x_1 - y_1 = -13 \quad \ldots\text{(i)}$$

Also, PN is perpendicular to AB.

$$\therefore \quad \frac{y_1 + 2}{x_1 - 1} \times 1 = -1$$



$$\Rightarrow \quad y_1 + 2 = -x_1 + 1$$

$$\Rightarrow \quad x_1 + y_1 = -1 \quad \ldots\text{(ii)}$$

On solving Eqs. (i) and (ii), we get

$$x_1 = -7, y_1 = 6$$

So, the coordinates of B are $(-7, 6)$.

Similarly, the coordinates of C are $\left(\frac{11}{5}, \frac{2}{5}\right)$.

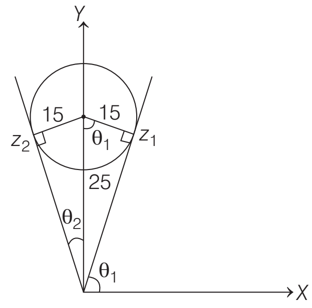
Hence, equation of BC is

$$y - 6 = \frac{\frac{2}{5} - 6}{\frac{11}{5} + 7}(x + 7)$$

$$\Rightarrow \quad y - 6 = -\frac{14}{23}(x + 7)$$

$$\Rightarrow \quad 14x + 23y - 40 = 0$$

16 *(b)* We have,



Clearly, $\max \text{amp}(z) = \text{amp}(z_2)$

and $\min \text{amp}(z) = \text{amp}(z_1)$

Now,

$$\text{amp}(z_1) = \theta_1 = \cos^{-1}\left(\frac{15}{25}\right) = \cos^{-1}\left(\frac{3}{5}\right)$$

and $$\text{amp}(z_2) = \frac{\pi}{2} + \theta_2 = \frac{\pi}{2} + \sin^{-1}\left(\frac{15}{25}\right)$$

$$= \frac{\pi}{2} + \sin^{-1}\left(\frac{3}{5}\right)$$

$$\therefore \quad |\max \text{amp}(z) - \min \text{amp}(z)|$$

$$= \left|\frac{\pi}{2} + \sin^{-1}\frac{3}{5} - \cos^{-1}\frac{3}{5}\right|$$

$$= \left|\frac{\pi}{2} + \frac{\pi}{2} - \cos^{-1}\frac{3}{5} - \cos^{-1}\frac{3}{5}\right|$$

$$= \pi - 2\cos^{-1}\left(\frac{3}{5}\right)$$

17 *(b)* Clearly, $\tan A = \frac{5}{12} = -\tan C$,

$$\cos B = -\frac{3}{5} = -\cos D$$

[$\because$ in cyclic quadrilateral, $A + C = \pi$ and $B + D = \pi$]

Now, $\tan C = -\frac{5}{12}$

$$\Rightarrow \quad \cos C = -\frac{12}{13} = \alpha \quad \text{(say)}$$

and $\cos D = \frac{3}{5} \Rightarrow \tan D = \frac{4}{3} = \beta$ (say)

$\therefore$ Required equation is

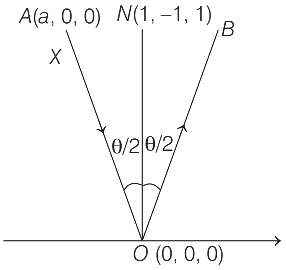
$$x^2 - (\alpha + \beta)x + \alpha\beta = 0$$

$$x^2 - \left(-\frac{12}{13} + \frac{4}{3}\right)x + \left(-\frac{12}{13} \times \frac{4}{3}\right) = 0$$

$$\Rightarrow \quad 39x^2 - 16x - 48 = 0$$

18 *(d)* Let the source of light be situated at $A(a, 0, 0)$, where $a \neq 0$.

Let AO be the incident ray and OB be the reflected ray, ON is the normal to the mirror at O.



Then, $\angle AON = \angle NOB = \dfrac{\theta}{2}$ (say)

DR's of **OA** are $(a, 0, 0)$ and so its **DC's** are (1, 0, 0).

DC's of **ON** are $\dfrac{1}{\sqrt3}, -\dfrac{1}{\sqrt3}, \dfrac{1}{\sqrt3}$

$\therefore \quad \cos\dfrac{\theta}{2} = \dfrac{1}{\sqrt3}$

Let **l** , **m** and **n** be the **DC's** of the reflected ray **OB**.

Then, $\dfrac{l+1}{2\cos\frac{\theta}{2}} = \dfrac{1}{\sqrt3}, \quad \dfrac{m+0}{2\cos\frac{\theta}{2}} = -\dfrac{1}{\sqrt3}$

and $\quad \dfrac{n+0}{2\cos\frac{\theta}{2}} = \dfrac{1}{\sqrt3}$

$\Rightarrow l = \dfrac{2}{3} - 1,$

$m = -\dfrac{2}{3}, n = \dfrac{2}{3}$

$\Rightarrow \quad l = -\dfrac{1}{3}, m = -\dfrac{2}{3}, n = \dfrac{2}{3}$

Hence, DC's of the reflected ray are $-\dfrac{1}{3}, -\dfrac{2}{3}, \dfrac{2}{3}.$

19 *(a)* Let $f(x) = 4x^2 - 2x + a$

Since, both roots of $f(x) = 0$ lie in the interval $(-1, 1)$, we can take

$D \ge 0, f(-1) > 0$ and $f(1) > 0$

1. Consider $D \ge 0$,

$\Rightarrow \quad (-2)^2 - 4\cdot 4\cdot a \ge 0$

$\Rightarrow \quad a \le \dfrac{1}{4}$ ...(i)

2. Consider $f(-1) > 0$,

$4(-1)^2 - 2(-1) + a > 0$

$\Rightarrow \quad a > -6$ ...(ii)

3. Consider $f(1) > 0$,

$4(1)^2 - 2(1) + a > 0$

$\Rightarrow \quad a > -2$ ...(iii)

From Eqs. (i), (ii) and (iii), we get

$-2 < a \le \dfrac{1}{4}$

20 *(c)* Since the system of given equation has a non-trivial solution, therefore

$$\begin{vmatrix} \lambda & \sin\theta & \cos\theta \\ 1 & \cos\theta & \sin\theta \\ -1 & \sin\theta & -\cos\theta \end{vmatrix} = 0$$

$\Rightarrow \lambda[-\cos^2\theta - \sin^2\theta]$
$-\sin\theta[-\cos\theta + \sin\theta]$
$+\cos\theta\,[\sin\theta + \cos\theta] = 0$

$\Rightarrow \quad -\lambda + \sin\theta\cos\theta - \sin^2\theta$
$+\sin\theta\cos\theta + \cos^2\theta = 0$

$\Rightarrow \quad -\lambda + 2\sin\theta\cos\theta + \cos^2\theta - \sin^2\theta = 0$

$\Rightarrow \quad -\lambda + \sin 2\theta + \cos 2\theta = 0$

$\Rightarrow \quad \lambda = \sqrt2\cos\left(2\theta - \dfrac{\pi}{4}\right)$ ...(i)

$\because \quad -1 \le \cos\left(2\theta - \dfrac{\pi}{4}\right) \le 1 \quad \forall \theta \in R$

$\therefore -\sqrt2 \le \sqrt2\cos\left(2\theta - \dfrac{\pi}{4}\right) \le \sqrt2 \;\; \forall\theta \in R$

$\Rightarrow \quad -\sqrt2 \le \lambda \le \sqrt2$ [using Eq. (i)]

Hence, $\lambda \in [-\sqrt2, \sqrt2]$.

21 *(3)* Let $I = \int_0^{\pi}[2\sin x]\,dx$

$= \int_0^{\pi/6}[2\sin x]\,dx + \int_{\pi/6}^{\pi/2}[2\sin x]\,dx$
$+\int_{\pi/2}^{5\pi/6}[2\sin x]\,dx + \int_{5\pi/6}^{\pi}[2\sin x]\,dx$

$= 0 + \int_{\pi/6}^{\pi/2} 1\,dx + \int_{\pi/2}^{5\pi/6} 1\,dx + 0$

$= [x]_{\pi/6}^{\pi/2} + [x]_{\pi/2}^{5\pi/6}$

$= \dfrac{\pi}{2} - \dfrac{\pi}{6} + \dfrac{5\pi}{6} - \dfrac{\pi}{2} = \dfrac{2\pi}{3} \quad \therefore \quad T = 3$

22 *(45)* Given, *A*'s are 30 sets with five elements each, so

$$\sum_{i=1}^{30} n(A_i) = 5 \times 30 = 150 \quad \text{...(i)}$$

If there are **m** distinct elements in **S** and each element of **S** belongs to exactly 10 of the A_i's, then

$$\sum_{i=1}^{30} n(A_i) = 10\,m \quad \text{...(ii)}$$

From Eqs.(i) and (ii), we get $m = 15$

Similarly, $\sum_{j=1}^{n} n(B_j) = 3\,n$

and $\quad \sum_{j=1}^{n} n(B_j) = 9\,m$

$\therefore \quad 3\,n = 9\,m$

$\Rightarrow \quad n = \dfrac{9\,m}{3} = 3 \times 15 = 45$

23 *(72)* Given, $p = 2q$

$\because \quad p + q = 1 \;\Rightarrow\; p = \dfrac{2}{3}$ and $q = \dfrac{1}{3}$

Now, required probability

$= {}^4C_2\left(\dfrac{2}{3}\right)^2\left(\dfrac{1}{3}\right)^2 + {}^4C_3\left(\dfrac{2}{3}\right)^3\left(\dfrac{1}{3}\right)^1$
$+ {}^4C_4\left(\dfrac{2}{3}\right)^4$

$= 6 \times \dfrac{4}{81} + \dfrac{4\times 8}{81} + \dfrac{1\times 16}{81}$

$= \dfrac{72}{81} = \dfrac{8}{9}$

$\therefore p = 8, q = 9$

Hence, $pq = 8 \times 9 = 72$

24 *(16)* Let the equation of plane be

$\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1.$

Let the centroid of tetrahedron **OABC** is (α, β, γ), then

$\alpha = \dfrac{0+a+0+0}{4}, \quad \beta = \dfrac{0+0+b+0}{4},$

$\gamma = \dfrac{0+0+0+c}{4}$

$\Rightarrow \quad a = 4\alpha, b = 4\beta, c = 4\gamma$

Since, distance of plane from origin is **P**,

$$P = \frac{1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}}$$

$\Rightarrow \quad \dfrac{1}{a^2} + \dfrac{1}{b^2} + \dfrac{1}{c^2} = \dfrac{1}{P^2}$

On putting the values of *a*, *b* and *c*, then

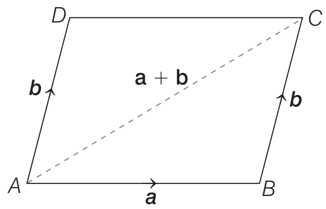
$\dfrac{1}{16\alpha^2} + \dfrac{1}{16\beta^2} + \dfrac{1}{16\gamma^2} = \dfrac{1}{P^2}$

Hence, the locus of centroid is

$\dfrac{1}{x^2} + \dfrac{1}{y^2} + \dfrac{1}{z^2} = \dfrac{16}{P^2}$

$\therefore \quad k^2 = 16$

25 *(4)* Clearly, $\quad AC = a + b$



$\Rightarrow |AC|^2 = |a|^2 + |b|^2 + 2a\cdot b$

$\Rightarrow |AC|^2 = \{|3\alpha - \beta|^2 + |\alpha + 3\beta|^2$
$+ 2(3\alpha - \beta)(\alpha + 3\beta)\}$

$= 9\alpha^2 + \beta^2 - 6\alpha\beta + \alpha^2 + 9\beta^2 + 6\alpha\cdot\beta$
$+ 6\alpha^2 - 6\beta^2 + 16\alpha\cdot\beta$

$\Rightarrow |AC|^2 = 16\alpha^2 + 4\beta^2 + 16\alpha.\beta$

$\Rightarrow |AC|^2 = 64 + 16 + 16\,|\alpha||\beta|\cos\dfrac{\pi}{3}$

$\Rightarrow |AC|^2 = 64 + 16 + 16 \times 2 \times 2 \times \dfrac{1}{2}$

$\Rightarrow |AC| = 4\sqrt7$

Similarly, $|BD| = |a - b| = 4\sqrt3$

$\therefore \quad p = 4$

26 *(4)* Clearly,

$$|A| = \begin{vmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{vmatrix}$$

$\Rightarrow \quad |A| = 2(1 + \sin^2\theta)$

Now, $0 \le \sin^2\theta \le 1$, for all $\theta \in [0, 2\pi]$.

$\Rightarrow \quad 2 \le 2 + 2\sin^2\theta \le 4$, for all $\theta \in [0, 2\pi]$

$\therefore$ Largest value of $|A|$ is 4.

27 *(2)* Let

$$A = \frac{(\sin^3 x + \cos^3 x) + (\cos^4 x - \sin^4 x)}{(1 + \cos x)(1 - \sin x)}$$

$$= \frac{(\sin^3 x + \cos^3 x) + (\cos x + \sin x)(\cos^2 x + \sin^2 x)(\cos x - \sin x)}{(1 + \cos x)(1 - \sin x)}$$

$$= \frac{(\sin x + \cos x)\{(1 - \sin x \cos x) + (\cos x - \sin x)\}}{1 + \cos x - \sin x - \sin x \cos x}$$

$\Rightarrow \quad A = \sin x + \cos x \qquad \ldots(i)$

$$= \frac{1}{\sqrt{2}} \sin\left(\frac{\pi}{4} + x\right)$$

Again from Eq. (i), we get

$$A = \frac{1}{\sqrt{2}} \cos\left(\frac{\pi}{4} - x\right)$$

$\therefore \quad a = 2$

28 *(288)* Let two particular boys as one boy, we have only four boys.

$\therefore$ 5 boys can be seated at a round table when two particular boys are always together $= 3!2!$

$\Rightarrow$ 4 girls have are 4 places.

$\therefore$ 4 girls can be arranged in 4! ways.

$\therefore$ Required number $= 3!2!4! = 6 \cdot 2 \cdot 24$

$= 288$

29 *(8)* $\therefore S_n = \dfrac{1\left(1 - \left(\frac{1}{2}\right)^n\right)}{\left(1 - \frac{1}{2}\right)} = 2 - \dfrac{1}{2^{n-1}}$

or $\quad 2 - S_n = \dfrac{1}{2^{n-1}} < \dfrac{1}{100}$

$\left[\because 2 - S_n < \dfrac{1}{100}\right]$

or $\quad 2^{n-1} > 100 > 2^6$

$\Rightarrow \quad 2^{n-1} > 2^6$

$\Rightarrow \quad n - 1 > 6$

$\therefore \quad n > 7$

Hence, least value of **n** is 8.

30 *(220)* We have,

$$P(X + Y = 3) = P(X = 0, Y = 3) + P(X = 1, Y = 2) + P(X = 2, Y = 1) + P(X = 3, Y = 0)$$

$$P(X = 0)P(Y = 3) + P(X = 1)P(Y = 2) + P(X = 2)P(Y = 1) + P(X = 3)P(Y = 0)$$

[$\because$ **X** and **Y** are independent]

$$= {}^5C_0\left(\frac{1}{2}\right)^5 \cdot {}^7C_3\left(\frac{1}{2}\right)^7 + {}^5C_1\left(\frac{1}{2}\right)^5 {}^7C_2\left(\frac{1}{2}\right)^7 + {}^5C_2\left(\frac{1}{2}\right)^5 {}^7C_2\left(\frac{1}{2}\right)^7 + {}^5C_3\left(\frac{1}{2}\right)^5 {}^7C_0\left(\frac{1}{2}\right)^7$$

$$= \left(\frac{1}{2}\right)^{12} [(1)(35) + (5)(21) + (10)(7) + (10)(1)]$$

$$= \frac{220}{2^{12}} = \frac{55}{1024} = \lambda \qquad \text{(given)}$$

$$\therefore 4096\lambda = 4096 \times \frac{55}{1024} = 4 \times 55 = 220$$